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Methods and applications of the mean-variance portfolio selection model

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METHODS AND APPLICATIONS OF THE MEAN-VARIANCE PORTFOLIO SELECTION MODEL

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ABSTRACT

The Mean-Variance portfolio selection model, or Efficient Market model, is examined in terms of the small investor. The performance is first tested on the small sample space of the thirty Dow Jones Industrials. The results show that it is possible to outperform the market by investing in the minimum-variance, or safest, portfolio. The Critical-Line algorithm as developed by Markowitz and modified by Sharpe is used in this analysis.

Since the Critical-Line algorithm is very time-consuming and does not always converge to a solution, an alternate algorithm is developed. This algorithm, referred to as the "Simplified Algorithm", is designed to find specific mean-variance efficient portfolios. It is shown that in the long run there is no significant difference in the performance of the portfolios calculated by the two algorithms.

The Simplified Algorithm is applied to the group of Institutional Growth Stocks and it is shown that the highest-expected-return portfolio substantially outperforms the market. This is in contrast to the results shown for the Dow Jones Industrials.

PREFACE

In recent years much attention has been given to the "Beta Theory" which has evolved out of Markowitz' pioneering work with the efficient market model. There have been a number of papers published involving case studies of mutual funds and large institutional portfolios. The general conclusion is that the portfolios fall near the efficient frontier and the larger-variance higher-expected-return growth portfolios tend to outperform the smaller-variance lower-expected-return income portfolios. The market portfolio can be found near the minimum-variance portfolio while the best performing growth portfolios have a beta of about 2.4.

A number of questions come immediately to our attention. For example, is the market average, as measured by the Dow Jones Industrial Average, a realistic figure to use in comparisons? We think not. Suppose the Dow Jones Industrial Average shows a 10% gain per year and the investor's growth portfolio shows a 14% gain per year. Does this mean that the investor has done a good job in selecting his portfolio? The answer could be no, because the market average for just growth stocks could show a 16% rate of return. Thus, we feel that if the investor's objective is growth, then his portfolio should at least perform as well as the market average for the top twenty or thirty growth stocks.

Likewise, if the objective is income, then the portfolio should do at least as well as the top rated income stocks.

Another problem that one faces is that of data collection. If all the securities listed on the big exchanges are candidates for any given portfolio, then several thousand securities must be analyzed. For the small investor and small researcher this is an impossible task. It is quite feasible, however, to collect data for 100 to 150 of the top rated stocks on a weekly basis.

In our research we prefer to follow only these top rated stocks. These stocks are split up into several groups. The efficient market model is tested on two of these groups, the Dow Jones Industrials and the Institutional Growth Stocks.

A number of other questions arise which are next to impossible to answer. The best that can be done is to try something that seems to work most of the time, yielding the best results. A case in point is that of determining the statistic for the expected rate of return for a particular security. Do you use a data base of daily price changes, or should it be based on weekly, monthly, or yearly price changes? Also should one use the logarithmic mean or the arithmetic mean? The same questions apply to the statistic used for the variance of the security. Experience has shown that price changes over subintervals of a one year period seem to yield statistics that work as well as any.

The big question that we would like to answer is this, does the efficient market model outperform the market? If so, what are the portfolio selection strategies? It will be shown that the model does outperform the market, assuming we have a reasonable statistic for the market. The important factor concerning the strategies to follow is that different classes of stocks require different portfolio selection strategies. It will be shown that if one is investing in income stocks or the Dow Jones Industrials the best strategy to follow is that of staying with the minimum-variance portfolio. The growth stocks, on the other hand, require the investor to stay with the highest expected return portfolio.

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I am especially grateful to Dr. C. Y. Ho, my major advisor, for his guidance, encouragement, and technical assistance in the preparation of this work.

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TABLES

THE EFFICIENT MARKET MODEL AND THE DOW JONES INDUSTRIALS

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This paper examines the performance of the Mean-Variance portfolio selection model when it is applied to the Dow Jones Industrials. It describes the statistics the model uses and how they are derived. Two strategies are used to test the performance of the model versus the performance of the market. In both cases the model outperforms the market. It is shown, contrary to the theory, that the minimum-variance portfolio, or low-risk portfolio, outperforms all other mean-variance efficient portfolios for this group of securities. Futhermore, the portfolios on the most inefficient frontier are shown to do as well as those on the efficient frontier. Thus, for securities of this type, it is suggested that

in the long run, the safest portfolio is the optimal growth portfolio.

There have been many papers written in recent years pertaining to the theoretical aspects of the portfolio selection problem. Recently Wagner and Lau [5] presented some evidence, based on simulated tests, that showed the rate of return on well-diversified, low-risk portfolios being significantly lower than the rate of return on higher-risk portfolios. This would tend to support the Mean-Variance school of thought. On the other hand, Hakansson [2] has presented some examples which show that the only portfolios that lead to ruin in the long run are Mean- Variance efficient. Thus Hakansson suggests that it is quite possible that the Mean-Variance model is severely compromised by the Capital Growth or Geometric Mean model.

It should be pointed out that most of the published work dealing with the Mean-Variance or Efficient Market model used a large sample space. That is, the portfolios analyzed contained anywhere from ten to 200 securities which were selected from the big stock exchanges. Mutual funds are prime examples of these large portfolios. It is the purpose of this paper to examine the performance of the Efficient Market model on a small sample space, namely, the 30 Dow Jones Industrials, to see how well the theory holds. This is a more realistic study as far as the small investor is concerned.

This paper will consider two primary questions. First, does the low-risk portfolio, as applied to the small sample space of the Dow Jones Industrials, yield a lower return than the higher-risk portfolios as suggested by Wagner and Lau? Secondly, are the Mean-Variance efficient portfolios outperformed by portfolios on the most inefficient frontier as one might infer from Hakansson's papers? Portfolios will be accumulated on the efficient frontier and it will be shown that it is <u>not</u> the higher-risk portfolios that yield higher returns. In fact, it is the <u>safest</u> portfolio that yields the greatest return. The portfolios on the efficient frontier will then be compared to those on the most inefficient frontier. The results will show that for this group of securities it is the variance that is of primary importance. Thus, it will be shown that the minimum-variance portfolio substantially outperforms the market average.

PORTFOLIO SELECTION MODEL

The model used in this paper follows the standard set of assumptions for the portfolio selection problem (see Hakansson [2]), except for the two premises about interest rates and short sales. This study did not allow any lending, borrowing, or short sales. The notation then reduces to the following:

x_j = the amount of investment capital at decision point j (the beginning of the j^{th} period)

e_{ij} = proceeds per unit of capital invested in opportunity i, where i = 1,2,....,30, in the j^{th} period

y_{ij} = the amount invested in opportunity i, where i = 1,2,....,30, at the beginning of the j^{th} period

$y^*_{ij}(x_j)$ = an optimal investment strategy for opportunity i, where i = 1,2,....,30, at decision point j

$$v_{ij} = \begin{cases} y_{ij}/x_j & x_j \neq 0 \\ 0 & x_j = 0 \end{cases} \qquad i = 1,2,\ldots,30$$

$$\bar{y}_j = (y_{1j},\ldots,y_{30j}) \qquad j = 1,2,\ldots$$

$$\bar{v}_j = (v_{1j},\ldots,v_{30j}) \qquad j = 1,2,\ldots$$

$$(\bar{v}_j) = \bar{v}_1,\ldots,\bar{v}_j \qquad j = 1,2,\ldots$$

As in the more general setting, v_{ij} denotes the proportion of capital x_j invested in opportunity i at the beginning of period j. The amount of capital at the end of period j now becomes:

$$x_{j+1} = \sum_{i=1}^{30} y_{ij} e_{ij} \qquad j = 1,2,\ldots$$

$$= x_j R_j(\bar{v}_j) \qquad j = 1,2,\ldots$$

where

$$R_j(\bar{v}_j) = \sum_{i=1}^{30} v_{ij} e_{ij} \qquad j = 1,2,\ldots$$

$R_j(\bar{v}_j)$ denotes one plus the return on the entire portfolio for period j. The problem is now to select the proportions $\bar{v}_j$ to produce the most favorable distribution of capital $R_j(\bar{v}_j)$ at the end of the period. This is accomplished by applying the critical-line algorithm as developed by Markowitz [3] and Sharpe [4].

The standard Mean-Variance approach is to

$$\text{Minimize } Z_j = -\lambda E(R_j(\overline{v}_j)) + Var(R_j(\overline{v}_j)) \qquad (1)$$
$$j = 1,2,\ldots.$$

where λ is the slope of the preference curve. To formulate Sharpe's diagonalized single index model let

$$e_{ij} = a_{ij} + b_{ij}I + c_{ij} \qquad i = 1,2,\ldots,30$$
$$j = 1,2,\ldots.$$

and

$$I = a_{31j} + c_{31j} \qquad j = 1,2,\ldots.$$

The basic assumptions are

$$E(c_{ij}) = 0 \qquad i = 1,2,\ldots,31$$
$$j = 1,2,\ldots.$$
$$Var(c_{ij}) = q_{ij} \qquad i = 1,2,\ldots,31$$
$$j = 1,2,\ldots.$$

and

$$Cov(c_{ij},c_{kj}) = 0 \qquad \text{for } i \neq k$$
$$j = 1,2,\ldots.$$

We now see that

$$R_j(\overline{v}_j) = \sum_{i=1}^{30} v_{ij}e_{ij} \qquad j = 1,2,\ldots.$$

$$= \sum_{i=1}^{30} v_{ij}(a_{ij} + b_{ij}I + c_{ij}) \qquad j = 1,2,\ldots.$$

$$= \sum_{i=1}^{30} v_{ij}(a_{ij} + c_{ij}) + \sum_{i=1}^{30} b_{ij}v_{ij}I$$
$$j = 1,2,\ldots.$$

$$= \sum_{i=1}^{31} v_{ij}(a_{ij} + c_{ij}) \qquad j = 1,2,\ldots.$$

where

$$v_{31j} = \sum_{i=1}^{30} b_{ij}v_{ij} \tag{2}$$

$$j = 1,2,\ldots.$$

Hence,

$$E(R_j(\bar{v}_j)) = \sum_{i=1}^{31} v_{ij}a_{ij} \tag{3}$$

$$j = 1,2,\ldots.$$

and

$$Var(R_j(\bar{v}_j)) = \sum_{i=1}^{31} v_{ij}^2 q_{ij} \tag{4}$$

$$j = 1,2,\ldots.$$

In order to assure some diversification a minimum of six securities were forced into the portfolios. This was easily done by placing bounds on v_{ij}. By substituting (3) and (4) into (1) and adding the constraints, Sharpe's model becomes

$$\text{Minimize } Z_j = -\lambda \sum_{i=1}^{31} v_{ij}a_{ij} + \sum_{i=1}^{31} v_{ij}^2 q_{ij}$$

Subject to

$$\sum_{i=1}^{30} v_{ij} = 1$$

$$\sum_{i=1}^{30} v_{ij}b_{ij} = v_{31j}$$

$$0 \le v_{ij} \le 1/6 \qquad i = 1,2,\ldots,30$$

$$j = 1,2,\ldots.$$

This model was tested on the Dow Jones Industrials over a twelve-year period beginning in December of 1959. This time period was subdivided into intervals of 40 trading days each. 40 trading days represent a period of about two months. The statistics were calculated using price changes over six subintervals. Longer periods of time were tried, but the best results seemed to occur when a one-year time span was used for the data base. Arithmetic percentage changes in prices were used and dividends were omitted. Geometric means were tried but there was no significant improvement in the results.

BUY-LOW-SELL-HIGH STRATEGY

The first test used a buy-low-sell-high strategy. Figure 1 shows the relative positions of the portfolios

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insert Figure 1 about here

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summarized in Table I and Table II. Table I lists the results for the four portfolios found on the efficient frontier. Notice that these portfolios are bounded away from

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insert Table I about here

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insert Table II about here

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the extreme outer boundary of the set of all feasible portfolios. This is due to the restriction placed on the minimum number of securities allowed in the portfolio.

Note that by timing the market accurately an investor would have been able to achieve a return of approximately 16.3% per year if he would have invested in the minimum-variance portfolio. The DJIA (Dow Jones Industrial Average) on the other hand, yielded approximately 10.9% per year. Thus, the minimum-variance portfolio outperforms the DJIA by better than 50%.

Table II shows the results of the four portfolios found on the most inefficient frontier. Notice how the rate of return increases as the variance decreases. It appears that it is the variance that affects the rate of return the most. Note also, that there need not be a large number of securities in the minimum-variance portfolio. The last column of Table I and Table II shows the results of rounding off the minimum-variance portfolio to the nearest six securities. In most cases the performance actually increased.

BUY-HOLD STRATEGY

The next test that was performed was that of a buy-and-hold strategy. The buying was done periodically in much the same way a person invests periodically in a mutual fund. One unit of capital was invested at the beginning of each period and a portfolio was accumulated.

Table III shows the results for this test. Once again

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insert Table III about here

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it is the minimum-variance portfolio that outperforms the others. The market (the Dow Jones Industrials weighted evenly in terms of capital invested) averaged about 3.6% gain per year versus about 4.2% for the minimum-variance portfolio. If dividends would have been included and reinvested these figures would have been larger.

CONCLUSION

The Sharpe model as applied to the 30 Dow Jones Industrials does not substantiate the claim that larger variance portfolios yield higher returns. For this group of stocks, during this time span, it was the minimum-variance, or safest, portfolio that yielded the greatest return. This portfolio substantially out-gained the market.

It is interesting to note that the portfolios on the most inefficient frontier yielded returns as great as those on the efficient frontier. This indicates that if one is considering a group of securities such as the Dow Jones Industrials, it is the variance of the portfolio that is of primary concern.

One very important factor that is quite apparent, is timing. The best portfolio using a buy-and-hold strategy yielded 4.2% gain per year, whereas if good timing could have been used, a gain of more than 16% could have been attained. Thus, one might conclude that a good procedure to follow would be to use the model in conjunction with a good timing factor.

It should not be concluded that these results would be the same for all classes of securities. The Dow Jones Industrials, for example, contain only two or three good growth stocks. Hence, it would be of interest to see how well the model would perform on other groups of securities. The Institutional Growth Stocks might be one such group.

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APPENDIX

Table IV shows the percentage gains for each of the 30 Dow Jones Industrials during the four major bull markets. These time periods are December 12, 1960 to November 24, 1961, July 17, 1962 to January 24, 1966, September 9, 1966 to November 27, 1968, and July 8, 1970 to April 22, 1971 respectively. The remaining tables show the composition of the various portfolios analyzed in this paper. All of the data in the last four tables are in terms of percentages as denoted by ($\bar{v}_j$). Thus, each portfolio should total to 100%.

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insert Table IV here

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insert Table V-A through
Table V-D here

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insert Table VI here

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insert Table VII-A through
Table VII-D here

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insert Table VIII here

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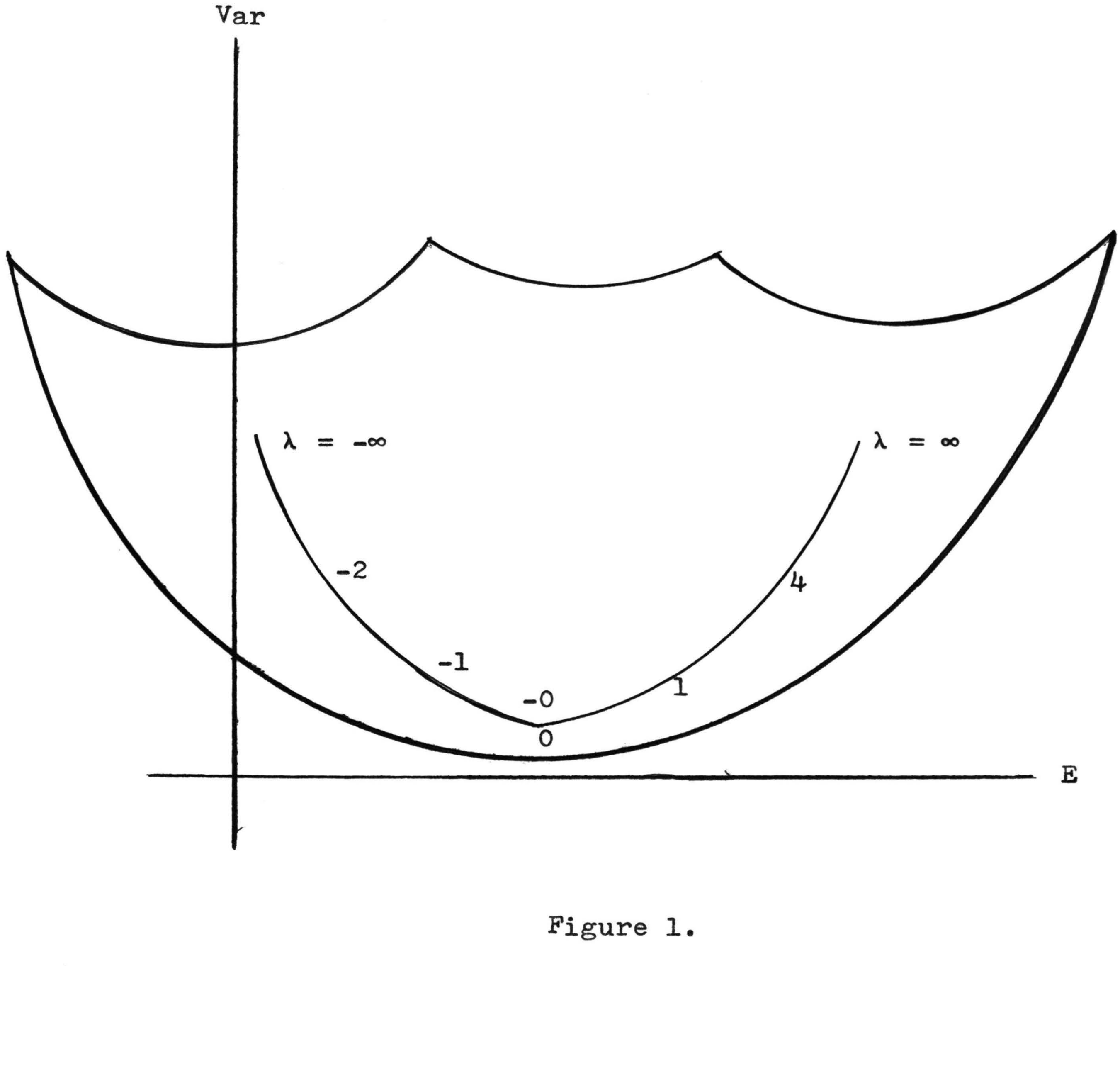

Figure 1.

TABLE I

PERIOD	DJIA	MARKET AVERAGE	$\lambda = \infty$	4	1	0	$\lambda = 0$ ROUNDED TO NEAREST SIX SECURITIES
I	13.71%	20.25%	43.13%	32.63%	35.89%	33.43%	34.03%
II	62.42	73.79	65.23	93.33	93.33	100.35	105.16
III	22.76	28.59	17.14	32.82	36.91	35.86	36.73
IV	40.08	41.35	33.74	43.83	47.07	52.80	58.11
GAIN PER YEAR	10.93	12.62	12.41	15.09	15.82	16.28	16.95

TABLE II

PERIOD	$\lambda =$ ∞	-2	-1	-0	λ = -0 ROUNDED TO NEAREST SIX SECURITIES
I	14.13%	21.88%	26.17%	29.66%	29.51%
II	52.79	87.43	101.43	107.09	101.55
III	32.02	24.02	30.78	35.86	36.73
IV	56.45	52.39	49.82	52.80	58.11
GAIN PER YEAR	12.14	13.87	15.24	16.34	16.44

TABLE III

λ =	VALUE OF PORTFOLIO	GAIN PER YEAR
∞	$75.17	2.90%
5	77.22	3.38
4	77.35	3.41
2	78.61	3.69
1	79.05	3.79
0	80.84	4.18
MARKET AVERAGE	78.20	3.60

TABLE IV

	PERIOD			
SECURITY	I	II	III	IV
ACD	2.88%	28.62%	3.52	71.32%
AA	-9.77	45.41	9.12	35.40
AC	49.61	31.83	16.93	18.21
T	40.58	7.74	12.41	19.88
AT	70.12	16.98	13.22	43.59
A	15.25	125.90	54.64	-0.55
BS	4.85	20.96	5.19	5.62
C	23.38	442.05	63.79	80.29
DD	36.69	77.20	0.00	31.42
EK	-3.54	155.03	40.81	33.06
GE	1.94	82.88	22.39	78.81
GF	39.59	12.37	30.97	9.34
GM	30.51	112.31	12.95	39.60
GT	26.90	43.28	24.48	45.74
HR	22.35	98.45	-5.10	28.41
N	34.62	67.03	22.63	19.46
IP	14.40	30.05	52.40	20.16
JM	6.82	25.29	72.49	42.39
OI	0.83	58.25	26.09	48.24
PG	44.16	-0.74	37.05	26.68

TABLE IV (continued)

	PERIOD			
SECURITY	I	II	III	IV
S	57.53%	85.34%	25.84%	64.24%
SD	15.93	46.22	24.51	61.36
J	20.51	59.62	30.41	50.35
SWX	-18.09	59.72	55.92	57.28
TX	32.30	56.04	41.60	52.45
UK	8.16	50.00	-6.65	42.01
UA	19.50	200.43	7.45	71.87
X	4.20	15.18	9.54	14.11
WX	-22.94	130.63	85.98	46.09
Z	38.33	29.73	67.28	83.76
AVERAGE	20.25	73.79	28.59	41.35
DJIA	13.71	62.42	22.76	40.08

TABLE V-A

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AA	6.14%	AA	1.19%	T	16.67%	AC	2.10%
AC	10.37	A	2.20	GF	8.66	AT	1.30
T	16.67	GM	16.67	JM	16.67	A	4.14
AT	16.67	JM	3.51	OI	7.49	BS	0.64
BS	6.64	S	16.67	PG	16.67	C	12.16
C	8.66	SD	16.67	S	16.67	DD	0.25
DD	6.51	TX	16.67	TX	16.03	EK	16.67
GM	5.17	UA	16.67	UK	1.15	GT	1.22
JM	0.49	WX	9.76			IP	3.58
SD	3.69					S	16.67
J	16.67					SD	16.67
SWX	1.44					SWX	16.67
Z	0.88					TX	5.12
						UA	2.83

TABLE V-B

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AC	11.96%	GM	16.67%	T	16.67%	AC	1.46%
T	16.67	S	16.67	GF	8.76	AT	8.30
AT	16.67	SD	16.67	JM	16.67	A	3.34
C	0.53	J	16.67	OI	11.26	DD	15.17
GM	4.37	TX	16.67	PG	16.67	EK	0.85
JM	6.08	UA	16.67	S	16.67	GE	16.67
PG	4.34			J	2.98	GF	0.24
SD	7.68			SWX	9.96	GM	1.72
J	16.67			TX	0.38	HR	0.37
SWX	4.71					JM	7.12
TX	0.57					PG	2.32
Z	9.76					SD	0.50
						J	16.67
						TX	16.67
						UK	5.78
						X	1.86
						WX	0.98

TABLE V-C

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
T	16.67%	GM	16.67%	AA	1.91%	AT	16.67%
AT	16.67	S	16.67	T	16.67	DD	16.67
GF	2.32	SD	16.67	EK	6.76	GE	4.07
JM	12.43	J	16.67	GT	2.11	GF	1.54
PG	16.67	TX	16.67	HR	0.36	JM	10.93
SD	7.33	UA	16.67	JM	16.67	PG	9.88
J	8.87			OI	16.67	J	14.13
SWX	9.20			PG	16.67	SWX	4.45
Z	9.86			S	16.67	TX	16.67
				UA	5.53	WX	4.99

TABLE V-D

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
T	16.67%	AC	16.67%	AA	16.67%	AT	16.67%
AT	16.67	GM	16.67	EK	16.67	GF	16.67
GF	16.67	S	16.67	GT	16.67	N	16.67
JM	16.67	SD	16.67	HR	16.67	PG	16.67
PG	16.67	J	16.67	OI	16.67	SWX	16.67
S	16.67	TX	16.67	UA	16.67	WX	16.67

TABLE VI

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AC	16.67%	GM	16.67%	T	16.67%	C	16.67%
T	16.67	S	16.67	GF	16.67	EK	16.67
AT	16.67	SD	16.67	JM	16.67	S	16.67
BS	16.67	TX	16.67	PG	16.67	SD	16.67
C	16.67	UA	16.67	S	16.67	SWX	16.67
J	16.67	WX	16.67	TX	16.67	TX	16.67

TABLE VII-A

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AC	7.27%	ACD	16.67%	T	16.67%	AC	2.10%
T	16.67	AA	5.00	GF	8.66	AT	1.30
AT	5.54	AC	6.68	JM	16.67	A	4.14
A	16.67	A	16.67	OI	7.49	BS	0.64
C	4.91	C	2.48	PG	16.67	C	12.16
DD	16.67	HR	7.19	S	16.67	DD	0.25
IP	16.67	JM	9.38	TX	16.03	EK	16.67
J	15.61	SWX	2.91	UK	1.15	GT	1.22
		UA	16.67			IP	3.58
		WX	16.37			S	16.67
						SD	16.67
						SWX	16.67
						TX	5.12
						UA	2.83

TABLE VII-B

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AA	2.74%	ACD	16.67%	ACD	1.48%	AC	2.50%
AC	6.72	AA	6.10	T	16.67	A	5.11
T	16.67	A	16.67	DD	7.19	BS	1.82
A	16.67	C	1.86	GF	8.41	DD	11.47
C	13.55	JM	14.13	JM	10.51	GE	16.67
DD	16.67	SWX	3.10	OI	4.29	HR	1.85
IP	16.67	UK	8.14	PG	16.67	JM	0.03
J	10.33	UA	16.67	S	16.67	SD	16.67
		WX	16.67	TX	16.67	J	16.67
				UK	1.46	TX	16.67
						UK	2.63
						UA	4.21
						X	3.72

TABLE VII-C

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AA	9.70%	ACD	16.67%	ACD	4.16%	AC	1.60%
AC	10.20	AA	5.03	T	16.67	A	4.84
T	0.26	A	16.67	DD	16.67	BS	1.34
A	16.67	JM	15.13	GF	7.54	C	1.97
C	16.67	SWX	0.74	JM	2.54	DD	10.81
DD	16.67	UK	16.67	OI	1.12	GE	16.67
IP	16.67	UA	12.44	PG	16.67	HR	1.56
J	13.17	WX	16.67	S	16.67	SD	16.67
				TX	16.67	J	16.67
				UK	1.30	TX	16.67
						UA	7.88
						X	3.33

TABLE VII-D

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AA	16.67%	ACD	16.67%	ACD	16.67%	ACD	16.67%
A	16.67	N	16.67	DD	16.67	BS	16.67
BS	16.67	JM	16.67	GE	16.67	C	16.67
C	16.67	UK	16.67	GM	16.67	OI	16.67
DD	16.67	X	16.67	WX	16.67	SD	16.67
IP	16.67	WX	16.67	Z	16.67	UA	16.67

TABLE VIII

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AC	16.67%	ACD	16.67%	T	16.67%	C	16.67%
T	16.67	A	16.67	GF	16.67	EK	16.67
A	16.67	HR	16.67	JM	16.67	S	16.67
DD	16.67	JM	16.67	PG	16.67	SD	16.67
IP	16.67	UA	16.67	S	16.67	SWX	16.67
J	16.67	WX	16.67	TX	16.67	TX	16.67

AN ALGORITHM FOR APPROXIMATING SPECIFIC MEAN-VARIANCE EFFICIENT PORTFOLIOS

by

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and

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This paper presents a simple algorithm for approximating specific portfolios on the efficient frontier. The results of the simplified algorithm are then compared to the results of the critical-line algorithm developed by Markowitz and Sharpe. In the long run there seems to be no significant difference between the results of these two algorithms. The main advantage for using the simplified algorithm is that it requires far less CPU time for execution.

In a recent paper by Ho and Marsh [2] it was shown that the minimum-variance portfolio outperformed the market during the past twelve years. The study analyzed the Dow Jones Industrials and applied the critical-line algorithm to the single index model as described by Sharpe [4]. Since the minimum-variance portfolio is the last portfolio selected by the critical-line algorithm, much time is spent calculating unwanted corner portfolios. For example, it is not uncommon to find twenty or more portfolios on the efficient frontier when thirty securities are used in the analysis.

It is the purpose of this paper to present a simple algorithm which, when given a value for λ, will calculate the desired portfolio directly. In particular, the minimum-variance portfolio, as calculated by this algorithm, will be compared to the minimum-variance portfolio as calculated by Ho and Marsh [2] using the critical-line algorithm. The results of both algorithms will be shown to be approximately the same. The big advantage of the simplified algorithm is that it will be much faster to execute.

PORTFOLIO SELECTION MODEL AND NOTATION

Sharpe's single index model is adapted to the thirty Dow Jones Industrials with the market index being the Dow Jones Industrial Average. The assumtions and notation are the same as those for the standard portfolio selection problem except for the two premises about interest rates and short sales. This study does not allow any lending, borrowing, or short sales. Under these conditions Sharpe's model becomes

$$\text{Minimize } Z_j = -\lambda \sum_{i=1}^{31} v_{ij} a_{ij} + \sum_{i=1}^{31} v_{ij}^2 q_{ij} \qquad (1)$$

Subject to

$$\sum_{i=1}^{30} v_{ij} = 1$$

$$\sum_{i=1}^{30} v_{ij} b_{ij} = v_{31j}$$

$$0 \leq L_{ij} \leq v_{ij} \leq U_{ij} \leq 1 \qquad i = 1,2,\ldots\ldots,30$$

where

x_j = the amount of investment capital at decision point j (the beginning of the j^{th} period)

e_{ij} = proceeds per unit of capital invested in opportunity i, where $i = 1,2,\ldots\ldots,30$, in the j^{th} period

y_{ij} = the amount invested in opportunity i, where $i = 1,2,\ldots\ldots,30$, at the beginning of the j^{th} period

$y^*_{ij}(x_j)$ = an optimal investment strategy for opportunity i, where $i = 1,2,\ldots\ldots,30$, at decision point j

$$v_{ij} = \begin{cases} y_{ij}/x_j & x_j \neq 0 \\ 0 & x_j = 0 \end{cases} \qquad i = 1,2,\ldots\ldots,30$$

$$\bar{y}_j = (y_{1j},\ldots\ldots,y_{30j})$$
$$\bar{v}_j = (v_{1j},\ldots\ldots,v_{30j})$$
$$(\bar{v}_j) = \bar{v}_1,\ldots\ldots,\bar{v}_j$$

where

$j = 1,2,\ldots\ldots$

The other parameters in the model come from

$$e_{ij} = a_{ij} + b_{ij}I + c_{ij} \qquad i = 1,2,\ldots\ldots,30$$

and

$$I = a_{31j} + c_{31j}$$

where

$$j = 1,2,\ldots.$$

The basic assumptions are

$$E(c_{ij}) = 0 \qquad i = 1,2,\ldots,31$$

$$Var(c_{ij}) = q_{ij} \qquad i = 1,2,\ldots,31$$

and

$$Cov(c_{ij},c_{kj}) = 0 \qquad \text{for } i \neq k$$

where

$$j = 1,2,\ldots.$$

SOLUTION USING LAGRANGE MULTIPLIERS

The critical-line algorithm, as developed by Markowitz [3], determines a value for λ at the same time the portfolio is calculated. In this paper it is desired to fix λ first and then find the associated portfolio. For the remainder of this paper λ will be replaced by a predetermined value $\bar{\lambda}$.

The solution of (1) is found by using Lagrange multipliers. By introducing the Lagrange multipliers λ_f and λ_1, (1) becomes

$$\text{Minimize } Z_j = -\bar{\lambda} \sum_{i=1}^{31} v_{ij}a_{ij} + \sum_{i=1}^{31} v_{ij}^2 q_{ij} - \lambda_{fj}\left(\sum_{i=1}^{30} v_{ij} - 1\right)$$

$$-\lambda_{1j}\left(\sum_{i=1}^{30} v_{ij}b_{ij} - v_{31j}\right) \qquad (2)$$

where

$$j = 1,2,\ldots.$$

A system of linear equations is obtained by equating the partial derivatives of Z_j with respect to the unknown variables to zero. The resulting system of equations is as follows

$$-\bar{\lambda}a_{ij} + 2v_{ij}q_{ij} - \lambda_{fj} - \lambda_{1j}b_{ij} = 0$$

$$i = 1,2,\ldots.,30$$

$$-\bar{\lambda}a_{31j} + 2v_{31j}q_{31j} + \lambda_{1j} = 0$$

$$\sum_{i=1}^{30} v_{ij} - 1 = 0$$

$$\sum_{i=1}^{30} v_{ij}b_{ij} - v_{31j} = 0$$

where

$$j = 1,2,\ldots.$$

The first 31 equations yield

$$v_{ij} = (\bar{\lambda}a_{ij} + \lambda_{fj} + \lambda_{1j}b_{ij})/2q_{ij} \qquad i = 1,2,\ldots.,30$$

$$(3)$$

$$v_{31j} = (\bar{\lambda}a_{31j} - \lambda_{1j})/2q_{31j}$$

where

$$j = 1,2,\ldots.$$

The last two equations can now be rewritten as

$$\lambda_{fj}\sum_{i=1}^{30} 1/2q_{ij} + \lambda_{1j}\sum_{i=1}^{30} b_{ij}/2q_{ij} = 1 - \lambda\sum_{i=1}^{30} a_{ij}/2q_{ij} \qquad (4)$$

and

$$\lambda_{fj}\sum_{i=1}^{30} b_{ij}/2q_{ij} + \lambda_{1j}\left(\sum_{i=1}^{30} b_{ij}^2/2q_{ij} + 1/2q_{31j}\right) =$$

$$\bar{\lambda}\left(a_{31j}/2q_{31j} - \sum_{i=1}^{30} a_{ij}/2q_{ij}\right) \qquad (5)$$

where

$j = 1,2,....$

Equation (5) can now be solved for λ_{1j}. The solution is

$$\lambda_{1j} = A/(\sum_{i=1}^{30} b_{ij}^2/q_{ij} + 1/q_{31j}) \tag{6}$$

where

$$A = \bar{\lambda}(a_{31j}/q_{31j} - \sum_{i=1}^{30} a_{ij}b_{ij}/q_{ij}) - \lambda_{fj}\sum_{i=1}^{30} b_{ij}/q_{ij}$$

and

$j = 1,2,....$

λ_{fj} is found by substituting (6) into (4) and solving. The solution is

$$\lambda_{fj} = (B + C)/D \tag{7}$$

where

$$B = (2 - \bar{\lambda}\sum_{i=1}^{30} a_{ij}/q_{ij})(\sum_{i=1}^{30} b_{ij}^2/q_{ij} + 1/q_{31j})$$

$$C = \bar{\lambda}\sum_{i=1}^{30} b_{ij}/q_{ij}(\sum_{i=1}^{30} a_{ij}b_{ij}/q_{ij} - a_{31j}/q_{31j})$$

$$D = \sum_{i=1}^{30} 1/q_{ij}(\sum_{i=1}^{30} b_{ij}^2/q_{ij} + 1/q_{31j}) - (\sum_{i=1}^{30} b_{ij}/q_{ij})^2$$

and

$j = 1,2,....$

The complete solution for (2) is given by equations (3), (6), and (7) using back substitution. This looks like a neat analytic solution, but a small problem is encountered upon application. The problem centers about the last constraint for the objective function (1). The lower and upper bounds, L_{ij} and U_{ij}, for v_{ij} were not taken into consideration when the solution was derived. It is quite possible to have negative values, as well as values greater than one, for some of the v_{ij} even though the other constraints are satisfied. Thus, it is necessary to develop a procedure which will put some bounds on v_{ij}. This will be accomplished by the following algorithm.

THE ALGORITHM

The algorithm is based on the premise that if $v_{ij} \leq L_{ij}$ then the final solution will contain $v_{ij} = L_{ij}$. Likewise, if $v_{ij} \geq U_{ij}$ then the solution will contain $v_{ij} = U_{ij}$. This premise does not seem unreasonable when thinking in terms of the heuristic nature of the critical-line algorithm. The critical-line algorithm puts bounds on v_{ij} in a similar manner.

The algorithm is described as follows

Step 1. Solve (2) using equations (3), (6), and (7).

Step 2. If $L_{ij} \leq v_{ij} \leq U_{ij}$ for all i, stop.

Step 3. If $v_{ij} \leq L_{ij}$, set $v_{ij} = L_{ij}$.
If $v_{ij} \geq U_{ij}$, set $v_{ij} = U_{ij}$.

Step 4. Go to Step 1.

Note that as the algorithm passes from Step 4 to Step 1 there will be fewer unknown variables in (2). Notice also that it is not necessary to execute DO loops in order to calculate the sums in (6) and (7) each time the algorithm passes Step 1. For example, if $v_{kj} \leq L_{kj}$ or $v_{kj} \geq U_{kj}$, the sum $\sum_{i=1}^{30} 1/q_{ij}$ is replaced by $\sum_{i=1}^{30} 1/q_{ij} - 1/q_{kj}$. The other sums are transformed similarly. Also, the 2 appearing in (7) is replaced by $2 - 2L_{kj}$ or $2 - 2U_{kj}$ depending on whether $v_{kj} \leq L_{kj}$ or $v_{kj} \geq U_{kj}$. The algorithm then reduces to the following

Step 1'. Evaluate (3), (6), and (7).

Step 2'. If $L_{ij} \leq v_{ij} \leq U_{ij}$ for all i, stop.

Step 3'. If $v_{kj} \leq L_{kj}$, replace 2 by $2 - 2L_{kj}$ in (7).
If $v_{kj} \geq U_{kj}$, replace 2 by $2 - 2U_{kj}$ in (7).

Replace $\sum_{i=1}^{30} 1/q_{ij}$ by $\sum_{i=1}^{30} 1/q_{ij} - 1/q_{kj}$.

Replace $\sum_{i=1}^{30} a_{ij}/q_{ij}$ by $\sum_{i=1}^{30} a_{ij}/q_{ij} - a_{kj}/q_{kj}$.

Replace $\sum_{i=1}^{30} a_{ij}b_{ij}/q_{ij}$ by

$$\sum_{i=1}^{30} a_{ij}b_{ij}/q_{ij} - a_{kj}b_{kj}/q_{kj}.$$

Replace $\sum_{i=1}^{30} b_{ij}/q_{ij}$ by $\sum_{i=1}^{30} b_{ij}/q_{ij} - b_{kj}/q_{kj}$.

Replace $\sum_{i=1}^{30} b_{ij}^2/q_{ij}$ by $\sum_{i=1}^{30} b_{ij}^2/q_{ij} - b_{kj}^2/q_{kj}$.

Step 4'. Go to Step 1.

In the case of thirty securities it usually takes fewer than five iterations to produce the minimum-variance portfolio. Table III in the appendix shows a typical sequence of iterates in producing a minimum-variance portfolio.

SIMPLIFIED ALGORITHM VERSUS CRITICAL-LINE ALGORITHM

The simplified algorithm was tested on the thirty Dow Jones Industrials over a twelve year period beginning in December of 1959. This time span was divided into periods of forty trading days each, forty trading days being approximately two months. The statistics were calculated using arithmetic percentage changes in prices over six subintervals or periods (about one year) and omitting dividends. The lower and upper bounds on v_{ij} were set at 0 and 1 respectively. Other bounds were tried yielding slightly different results.

BUY-LOW-SELL-HIGH STRATEGY

Table I shows the results of the simplified algorithm using a buy-low-sell-high strategy. The results are com-

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insert Table I here

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pared to the Dow Jones Industrial Average, the market average (the Dow Jones Industrials weighted evenly in terms of capital), and the minimum-variance portfolio calculated by the critical-line algorithm. These portfolios can be found in Table V and Table VI of the appendix. Notice that the performances of both minimum-variance portfolios are about the same in the long run.

BUY-HOLD STRATEGY

Table II shows the results of a buy-and-hold strategy. Notice that the simplified algorithm actually outperforms the critical-line algorithm in the long run if the lower and

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insert Table II here

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upper bounds on v_{ij} are set at 0 and 1. As indicated earlier, the results were slightly different when the bounds were changed. In the case where the lower and upper bounds are set at 0 and 1/6 respectively, there is only a slight difference in the long run result (see Table II).

CONCLUSION

This paper presented a simplified algorithm for finding specific mean-variance portfolios. The long run results compared very favorably to those of the critical-line algorithm. The comparisons were made using the minimum-variance portfolios since they are in the area where the largest discrepancies occur. As Table II illustrates, the simplified algorithm does very well in the long run. Other portfolios do even better.

The big advantage in using the simplified algorithm is that less computer time is required for execution. This algorithm was run on the Control Data 6400 KRONOS time sharing system. It took just under ten seconds of CPU time to find all of the minimum-variance portfolios for the twelve-year test period. The critical-line algorithm was run on the IBM 360 Model 65 and it took well over five minutes to find the same set of portfolios.

ACKNOWLEDGMENT

The computing time was furnished by the University of Nebraska at Omaha, Omaha, Nebraska. The data was supplied by Wiesenberger Service, Incorporated, of New York, for whom Dr. Chung Y. Ho is a consultant.

APPENDIX

The four major bull markets that this papers uses are December 12, 1960 to November 24, 1961, July 17, 1962 to January 24, 1966, September 9, 1966 to November 27, 1968, and July 8, 1970 to April 22, 1971 respectively. Table III shows the portfolios calculated on each iteration of the simplified algorithm for the minimum-variance portfolio during the first time period analyzed. Table IV shows the percentage gains for each of the thirty Dow Jones Industrials during the four major bull markets. Table V shows the four portfolios calculated by the critical-line algorithm at the beginning of the four major periods. Table VI shows the four portfolios calculated by the simplified algorithm for the same major periods. All of the data in the last two tables are in terms of percentages as denoted by $(\bar{v}_j)$. Thus, each portfolio should total to 100%.

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insert Table III here

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insert Table IV here

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insert Table V here

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insert Table VI here

REFERENCES

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TABLE I

PERIOD	DJIA	MARKET	MINIMUM-VARIANCE PORTFOLIOS SIMPLIFIED ALGORITHM	CRITICAL-LINE ALGORITHM
I	13.71%	20.25%	39.70%	33.43%
II	62.42	73.79	105.74	100.35
III	22.76	28.59	25.66	35.86
IV	40.08	41.35	52.58	52.80
GAIN PER YEAR	10.93	12.64	16.22	16.28

TABLE II

CRITICAL-LINE ALGORITHM WITH LOWER AND UPPER BOUNDS ON v_{ij} OF 0 AND 1/6 RESPECTIVELY

λ =	VALUE OF PORTFOLIO	GAIN PER YEAR
∞	$75.17	2.90%
5	77.22	3.38
4	77.35	3.41
2	78.61	3.69
1	79.05	3.79
0	80.84	4.18

SIMPLIFIED ALGORITHM WITH LOWER AND UPPER BOUNDS ON v_{ij} OF 0 AND 1/6 RESPECTIVELY

0	80.79	4.18

SIMPLIFIED ALGORITHM WITH LOWER AND UPPER BOUNDS ON v_{ij} OF 0 AND 1.0 RESPECTIVELY

0	83.84	4.72

MARKET AVERAGE

	78.20	3.60

TABLE III

STOCK	ITERATIONS I	II	III	IV
ACD	-2.49%	0.00%	0.00%	0.00%
AA	3.85	3.83	3.71	3.70
AC	5.43	5.84	5.85	5.85
T	23.40	27.19	28.13	28.19
AT	24.19	25.03	24.70	24.66
A	1.30	0.73	0.50	0.48
BS	6.25	5.37	4.83	4.78
C	4.63	4.93	4.92	4.92
DD	14.90	10.43	8.14	7.94
EK	-0.63	0.00	0.00	0.00
GE	-1.11	0.00	0.00	0.00
GF	-1.39	0.00	0.00	0.00
GM	3.54	3.40	3.24	3.23
GT	0.88	0.11	-0.24	0.00
HR	0.21	-0.67	0.00	0.00
N	-1.08	0.00	0.00	0.00
IP	2.22	-0.70	0.00	0.00
JM	1.08	0.76	0.60	0.58
OI	0.47	-0.39	0.00	0.00

TABLE III (continued)

ITERATIONS

STOCK	I	II	III	IV
PG	-1.70	0.00	0.00	0.00
S	-3.25	0.00	0.00	0.00
SD	2.31	2.30	2.23	2.22
J	10.35	11.19	11.25	11.25
SWX	0.95	0.92	0.89	0.88
TX	1.09	0.28	-0.10	0.00
UK	1.67	-1.37	0.00	0.00
UA	0.07	-0.21	0.00	0.00
X	0.96	-0.79	0.00	0.00
WX	-0.73	0.00	0.00	0.00
Z	2.64	1.78	1.34	1.30

TABLE IV

	PERIOD			
STOCK	I	II	III	IV
ACD	2.88%	28.62%	3.52%	71.32%
AA	-9.77	45.41	9.12	35.40
AC	49.61	31.83	16.93	18.21
T	40.58	7.74	12.41	19.88
AT	70.12	16.98	13.22	43.59
A	15.25	125.90	54.64	-0.55
BS	4.85	20.96	5.13	5.62
C	23.38	442.05	63.79	80.29
DD	36.69	77.20	0.00	31.42
EK	-3.54	155.03	40.81	33.06
GE	1.94	82.88	22.39	78.81
GF	39.59	12.37	30.97	9.34
GM	30.51	112.31	12.95	39.60
GT	26.90	43.28	24.48	45.74
HR	22.35	98.45	-5.10	28.41
N	34.62	67.03	22.63	19.46
IP	14.40	30.05	52.40	20.16
JM	6.82	25.29	72.49	42.39
OI	0.83	58.25	26.09	48.24

TABLE IV (continued)

STOCK	I	II	III	IV
		PERIOD		
PG	44.16	-0.74	37.05	26.68
S	57.53	85.34	25.84	64.24
SD	15.93	46.22	24.51	61.36
J	20.51	59.62	30.41	50.35
SWX	-18.09	59.72	55.92	57.28
TX	32.30	56.04	41.60	52.45
UK	8.16	50.00	-6.65	42.01
UA	19.50	200.43	7.45	71.87
X	4.20	15.18	9.54	14.11
WX	-22.94	130.63	85.98	46.09
Z	38.33	29.73	67.28	83.76
MARKET AVERAGE				
	20.25	73.79	28.59	41.35
DJIA				
	13.71	62.42	22.76	40.08

TABLE V

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AA	6.14%	AA	1.29%	T	16.67%	AC	2.10%
AC	10.37	A	2.20	GF	8.66	AT	1.30
T	16.67	GM	16.67	JM	16.67	A	4.14
AT	16.67	JM	3.51	OI	7.49	BS	0.64
BS	6.64	S	16.67	PG	16.67	C	12.16
C	8.66	SD	16.67	S	16.67	DD	0.25
DD	6.51	TX	16.67	TX	16.03	EK	16.67
GM	5.17	UA	16.67	UK	1.15	GT	1.22
JM	0.49	WX	9.76			IP	3.58
SD	3.69					S	16.67
J	16.67					SD	16.67
SWX	1.44					SWX	16.67
Z	0.88					TX	5.12
						UA	2.83

TABLE VI

PERIOD I		PERIOD II		PERIOD III		PERIOD IV	
AA	3.70%	ACD	20.47%	T	40.37%	AC	1.10%
AC	5.85	AA	4.72	GF	2.33	AT	0.66
T	28.19	AC	5.82	JM	6.98	A	2.18
AT	24.66	A	18.79	OI	3.55	BS	0.32
A	0.48	C	2.30	PG	14.36	DD	6.45
BS	4.78	HR	3.59	S	30.90	EK	0.12
C	4.92	JM	8.90	TX	1.51	GE	13.30
DD	7.94	SWX	2.65			HR	0.64
GM	3.23	UA	17.15			JM	1.88
JM	0.58	WX	15.60			SD	12.87
SD	2.22					J	10.22
J	11.25					TX	46.14
SWX	0.88					UK	2.65
Z	1.30					X	1.49

THE EFFICIENT MARKET MODEL AND THE INSTITUTIONAL GROWTH STOCKS

by

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This paper examines the performance of the Mean-Variance portfolio selection model as it is applied to the Institutional Growth Stocks. It describes the statistics used in the model as well as the model itself. Two investment strategies are used to test the model. The performance of the model is compared to the performance of the market and it is shown that the model outperforms the market by a substantial margin. The portfolio that grows the fastest in the long run is the highest expected return portfolio.

In a recent paper by Ho and Marsh [2] it was shown that the minimum-variance portfolio was the optimal growth portfolio and it outperformed the market during the past twelve years. However, the study analyzed only those securities which make up the Dow Jones Industrial Average. Since there are very few growth stocks among the Dow Jones Industrials, one might ask if the same conclusion is valid for a group of growth-oriented stocks.

It is the purpose of this paper to examine the Mean-Variance model in terms of the Institutional Growth Stocks. It will be shown that it is the highest-expected-return portfolio that substantially outgains the other portfolios, including the market portfolio. This conclusion is just the opposite of that cited above. This would indicate that an investor's investment strategy might vary depending on the investment objective and the characteristics of the securities under consideration.

The growth stocks that will be studied form the twenty-one highest quality stocks with the highest rate of return during the past five years. This group, hereafter referred to as the Institutional Growth Stock Index, or IGSI, was selected by Fleming-Berger-Kent of Denver for Weisenberger Service, Incorporated, of New York. These stocks are listed in Table I.

PORTFOLIO SELECTION MODEL AND NOTATION

Sharpe's single index model is adapted to the twenty-one growth stocks with the composite average, IGSI, being used for the index of the model. The assumptions and notation are the same as those for the standard portfolio

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insert Table I about here

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selection problem except for the two premises about interest rates and short sales. This study did not allow any lending, borrowing, or short sales. Under these conditions Sharp's model becomes

$$\text{Minimize } Zj = -\lambda \sum_{i=1}^{22} v_{ij}a_{ij} + \sum_{i=1}^{22} v_{ij}^2 q_{ij} \qquad (1)$$

Subject to

$$\sum_{i=1}^{21} v_{ij} = 1$$

$$\sum_{i=1}^{21} v_{ij}b_{ij} = v_{22j}$$

$$0 \le L_{ij} \le v_{ij} \le U_{ij} \le 1 \qquad i = 1,2,\ldots.,21$$

$$j = 1,2,\ldots.$$

where

x_j = the amount of investment capital at decision point j (the beginning of the j^{th} period)

e_{ij} = proceeds per unit of capital invested in opportunity i, where i = 1,2,....,21, in the j^{th} period

y_{ij} = the amount invested in opportunity i, where i = 1, 2,....,21, at the beginning of the j^{th} period

$y^*_{ij}(x_j)$ = an optimal investment strategy for opportunity i, where i = 1,2,....,21, at decision point j

$$v_{ij} = \begin{cases} y_{ij}/x_j & x_j \neq 0 \\ 0 & x_j = 0 \end{cases} \qquad i = 1,2,\ldots\ldots,21$$

$$\overline{y}_j = (y_{1j},\ldots\ldots,y_{21j})$$

$$\overline{v}_j = (v_{1j},\ldots\ldots,v_{21j})$$

$$(\overline{v}_j) = \overline{v}_1,\ldots\ldots,\overline{v}_j$$

where

$$j = 1,2,\ldots\ldots$$

The other parameters in the model come from the regression equation

$$e_{ij} = a_{ij} + b_{ij}I + c_{ij} \qquad i = 1,2,\ldots\ldots,21$$

and

$$I = a_{22j} + c_{22j}$$

where

$$j = 1,2,\ldots\ldots$$

The basic assumptions are

$$E(c_{ij}) = 0 \qquad i = 1,2,\ldots\ldots,22$$

$$\mathrm{Var}(c_{ij}) = q_{ij} \qquad i = 1,2,\ldots\ldots,22$$

and

$$\mathrm{Cov}(c_{ij},c_{kj}) = 0 \qquad \text{for } i \neq k$$

where

$$j = 1,2,\ldots\ldots$$

The solution for (1) is found using the simplified algorithm developed by Marsh and Ho [4]. This algorithm executes much faster than the critical-line algorithm developed by Markowitz [3] and presented by Sharpe [5]. The highest-expected return portfolio is identical for both algorithms. When differences do occur, they are more pronounced in the region of the minimum-variance portfolio. In the long run, however, these differences are not significant.

The statistics used in (1) are generated using the relative price movements over a one year period. These statistics are in terms of arithmetic percentages and they are based on the average changes in prices over two-month subintervals. Since most of these growth stocks pay a very small dividend, dividends were excluded in this study. Other statistics were used in various comparisons, but experience has shown that in the long run they do not yield better results.

BUY-HOLD STRATEGY

The first test that is performed is that of a buy-and-hold strategy. The buying is done periodically in much the same way a person invests periodically in a mutual fund. One unit of capital is invested at the beginning of each period and a portfolio is accumulated. The test period runs from January 5, 1966 to May 18, 1973. This time period is divided into 48 subintervals of two months each. The first six periods are used for the data base for the first portfolio generated by the model. Thus, a total of 42 units of capital is invested in the accumulated portfolio.

----

insert Table II about here

The results of this experiment are shown in Table II. Notice that it is the highest-expected return portfolio that shows the most growth. By investing one unit of capital, represented here by $1.00, in the security that had the largest average gain for the previous year an investor would have ended up with a portfolio worth $96.95 at the end of 40 investing periods. This translates into a gain of about 23.5% per year. The last two periods occured in a declining market, hence, the portfolio shrank to $87.96 for a return of about 19.0% per year. In either case this is better than the market portfolio of $82.17 and $77.45 for returns of 19.4% and 15.9% respectively. Thus, the model outperforms the market by at least 20%.

Since DNB, LLY, and LZ were not listed on the New York Stock Exchange at the beginning of the test period, the test

----

insert Table III about here

----

was repeated deleting these securities. The results of this experiment are shown in Table III and the results are approximately the same.

BUY-LOW-SELL-HIGH STRATEGY

The next test that is performed is that of a buy-low-sell-high strategy. Table IV shows the results of getting in and out of the market three times during this time span. In each case the portfolio consists of the five stocks that had the largest average return for the previous year. The three portfolios gained 45.0%, 66.9%, and 175.7% during the three periods. This represents an accumulated gain of 566.9% in less than six years. The corresponding values for the market portfolio, which is the twenty-one stocks weighted evenly in dollar amounts, and the IGSI are 381.6% and 300.5% respectively. Note that the high-expected-return portfolio substantially outperforms the market. Its average gain per year being in the neighborhood of 40.0%, depending on how one would compound the gains.

The results of this strategy as applied to the minimum-variance portfolio are not shown in this paper, but they would show this portfolio lagging the market. Recall that for the buy-and-hold strategy, this portfolio, as shown in Table II and Table III, is the poorest performer.

CONCLUSION

This paper presented some evidence which indicates that the strategy to follow in selecting growth stock portfolios is that of selecting the highest-expected-yield stocks. Not only that, but the best performance occurred when fewer than five stocks were purchased at any given time. The fact that it is the highest-expected-return portfolio is quite significant, since it was this portfolio that showed the poorest performance when applied to the Dow Jones Industrials. The implication is clear; different investment objectives require different portfolio-selection policies.

Also, as in [1], timing is shown to be a very important factor. Substantial profits can be realized by forecasting the market accurately. Since accurate timing, or lack of it, can influence a portfolio's performance so radically, some of the other portfolio-selection techniques, such as the Mean-Variance approach, have not received as much attention as perhaps they deserve. It has been argued that there is no way to consistently do better than the market average. The evidence presented in this paper and in [1] shows that it is possible to improve on the market performance by as much as 20% or more. Thus, the best strategy for an investor to follow is to use the efficient market model in conjunction with a good timing factor.

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TABLE I

XRX	Xerox
JNJ	Johnson & Johnson
SYP	Simplicity Patterns
DOC	Dr. Pepper
AVP	Avon Products
SGP	Schering-Plough
EK	Eastman Kodak
LLY	Lilly (Eli)
LZ	Lubrizol
WLA	Warner-Lambert
IBM	Int'l. Business Machines
AMP	AMP
KO	Coca-Cola
MRK	Merck
IFF	Int'l. Flavors & Fragrances
MMM	Minnesota Mining & Mft.
PG	Procter & Gamble
PFE	Pfizer
S	Sears, Roebuck
AHP	American Home Products
DNB	Dun & Bradstreet

TABLE II

U_i (for all i)	λ	Ending Period #46	Ending Period #48
1.00	10000	$96.95	$87.96
.75	10000	96.33	87.15
.50	10000	95.72	86.35
.25	10000	91.03	82.66
.10	10000	84.83	79.14
.05	10000	82.39	77.72
1.00	10	92.51	83.58
.75	10	95.11	86.14
.50	10	96.00	86.43
.25	10	91.24	82.83
.10	10	84.83	79.14
.05	10	82.39	77.72
1.00	5	77.63	73.01
.75	5	89.18	82.08
.50	5	85.90	78.77
.25	5	88.56	80.27
.10	5	84.83	83.14
.05	5	82.39	77.72

TABLE II (continued)

U_i (for all i)	λ	Ending Period #46	Ending Period #48
1.00	2	$80.68	$74.78
.75	2	78.74	73.35
.50	2	80.90	75.73
.25	2	80.79	75.77
.10	2	84.83	79.14
.05	2	82.39	77.72
1.00	0	79.79	74.35
.75	0	77.59	74.47
.50	0	78.88	73.37
.25	0	78.73	73.44
.10	0	84.83	79.14
.05	0	82.39	77.72
Market Portfolio		$82.17	$77.45

TABLE III

U_i (for all i)	λ	Ending Period #46	Ending Period #48
1.00	10000	$95.48	$87.15
.75	10000	97.16	87.66
.60	10000	98.17	87.97
.50	10000	98.84	88.18
.40	10000	98.32	87.79
.33	10000	97.97	87.54
.25	10000	95.59	86.69
.20	10000	95.16	87.04
.10	10000	86.71	80.84
1.00	10	90.79	82.78
.75	10	96.04	86.62
.50	10	98.60	88.04
.33	10	97.97	87.47
.25	10	96.04	86.95
.20	10	94.13	85.98
.10	10	86.71	80.84
1.00	5	79.44	74.72
.75	5	89.87	82.39
.50	5	91.90	83.78
.33	5	92.99	83.78
.20	5	92.39	84.46

TABLE III (continued)

U_i (for all i)	λ	Ending Period #46	Ending Period #48
.10	5	$86.71	$80.84
1.00	2	81.34	74.88
.75	2	79.25	73.11
.50	2	81.16	75.44
.33	2	81.22	75.65
.20	2	81.88	76.22
.17	2	82.44	76.48
.10	2	86.71	80.84
1.00	0	80.19	74.11
.75	0	80.09	73.91
.50	0	79.87	73.63
.25	0	79.05	73.03
.10	0	86.71	80.84
.08	0	85.58	80.11
.07	0	85.29	79.87
Market Portfolio		$84.56	$79.02

TABLE IV

Period #1	Dec. 9, 1966	Jan. 5, 1968	% Gain
AMP	$28.125	$36.125	24.44%
AVP	43.500	65.375	51.45
IFF	19.875	37.875	90.56
SGP	27.125	33.750	24.42
XRX	71.125	92.375	29.87
Portfolio			44.95
Market Portfolio			39.05
IGSI			34.08

Period #2	Mar. 1, 1968	Oct. 24, 1969	% Gain
AVP	$ 58.375	$ 85.625	46.68%
DOC	4.375	7.750	77.14
IBM	230.750	291.750	26.42
LZ	18.500	31.250	68.91
SYP	13.125	28.250	115.23
Portfolio			66.88
Market Portfolio			57.00
IGSI			42.40

TABLE IV (continued)

Period #3	Jun. 5, 1970	Nov. 17, 1972	% Gain
DOC	$ 8.500	$ 28.500	235.29%
IFF	33.500	91.500	173.13
JNJ	43.500	129.375	197.41
LLY	39.375	74.750	89.84
SGP	49.125	139.000	182.95
Portfolio			175.72
Market Portfolio			120.61
IGSI			109.77

APPENDIX A

COMPUTER PROGRAM FOR SIMPLIFIED MODEL

The following computer program was written to implement the simplified algorithm used in this thesis. It uses the FORTRAN language on the Control Data 6400 KRONOS time-sharing system. This program can be easily modified to suit other computer systems.

```
      PRØGRAM SS(INPUT,ØUTPUT,TAPE1,TAPE2,TAPE3)
      DIMENSIØN STØCK(30),DJ(74),P(74,30),PØRT(30),MPØRT(30),
     +X(31),SD(30),BETA(30),ALPHA(30),TEMP(6),DATA(6,30),
     +INDEX(6),P1(73,30),E(30)
      REAL INDEX,MPØRT
      CALL GET(5HTAPE1,7HGSDATA,0,0)
      REWIND 1
      CALL GET(5HTAPE3,3HDØW,0,0)
      REWIND 3
      CALL GET(5HTAPE2,7HGSDATA2,0,0)
      REWIND 2
C  IØP = 0 DELETES PRINT OF STATISTICS
C  IØP = 1 PRINTS STATISTICS
C  IØP1 = 0 DELETES PRINT ØF LAMBDAS
C  IØP1 = 1 PRINTS LAMBDAS
C  IØP2 = 0 DELETES PRINT ØF PØRTFØLIØS
C  IØP2 = 1 PRINTS PØRTFØLIØS
C  ALBAR = THE PREDETERMINED VALUE FØR LAMBDA
```

```
C   AHIGH = UPPER BØUNDS FØR V(I,J)
C   LØWER BØUNDS FØR V(I,J) ARE ALL 0
C   IGAMMA = 0 USES BETA
C   IGAMMA = 1 USES MØDIFIED BETA
C   IØP3 = 0 USES GRØWTH STØCKS AS INDEX
C   IØP3 = 1 USES DØW AS INDEX
C   NØ = NUMBER ØF STØCKS
C   NP = NUMBER ØF PERIØDS USED IN ANALYSIS
C   ND = NUMBER ØF PERIØDS USED IN DATA BASE
C   MM = THE STARTING PERIØD
    READ,IGAMMA,IØP3
    READ,NØ,NP,ND,MM,ALBAR,IØP,IØP1,IØP2,AHIGH
    DØ 13 I=1,NØ
    PØRT(I)=0
    13 MPØRT(I) = 0
    NNØ=NØ+1
    DØ 14 I=1,ND
    14 INDEX(I)=0
    ALØW=0
    READ(1,100) (STØCK(I),I=1,NØ)
    NPP=NP+1
    READ(3,602) (DJ(I),I=1,NPP)
    602 FØRMAT (F10.3)
    100 FØRMAT (10A4)
    READ(1,101) ((P(I,J),I=1,NPP),J=1,NØ)
    NPPN=NPP
```

```
READ(2,101) ((P1(I,J),I=1,NPPN),J=1,NØ)
500 FØRMAT (/,5(6F9.3,/))
101 FORMAT (F8.3)
MN=NP-ND
DØ 11 L=MM,MN
DØ 20 I=1,NØ
20 P(L,I)=P1(L,I)
AK=1
L1=L+ND-1
L2=L1+1
IF(IØP3.EQ.1) GØ TØ 1
DØ 601 I=L,L2
DJ(I)=0
DØ 601 J=1,NØ
601 DJ(I)=DJ(I)+P(I,J)
1 CØNTINUE
DØ 2 I=L,L1
II=I-L+1
DØ 3 J = 1,NØ
X(J)=0.1
DATA(II,J)=(P(I+1,J)-P(I,J))*100/P(I,J)
3 CØNTINUE
2 INDEX(II)=(DJ(I+1)-DJ(I))*100/DJ(I)
AM=0
DØ 7 I=1,ND
7 AM=AM+INDEX(I)
```

```
AM=AM/ND
VAR=0
DØ 8 I=1,ND
8 VAR=VAR+(INDEX(I)-AM)**2
VAR=VAR/ND
IF (IØP.NE.1) GØ TØ 303
PRINT 302
302 FORMAT (*                ALPHA      BETA       VAR        E*)
303 CØNTINUE
EEE=0
DØ 4 J=1,NØ
DØ 5 I=1,ND
TEMP(I)=DATA(I,J)
IF (IGAMMA.EQ.1) GO TO 5
IF (INDEX(I).LT.).)) TEMP(I)=-TEMP(I)
5 CØNTINUE
CALL REGRESS (TEMP,INDEX,ND,ALPHA(J),BETA(J),AM,SD(J),AVV)
SD(J)=SD(J)**2
E(J)=ALPHA(J)+BETA(J)*AM
IF (IØP.NE.L) GØ TØ 4
EEE=E(J)/NØ+EEE
PRINT 300,STØCK(J),ALPHA(J),BETA(J),SD(J),E(J)
300 FORMAT (A4,4F12.4)
4 CØNTINUE
IF (IØP.EQ.1) PRINT 409,AM,VAR,EEE
409 FØRMAT (4X,F12.4,F24.4,F12.4)
```

```
SUMQ=0
SUMA=0
SUMBQ=0
SUMB2Q=0
SUMAB=0
DØ 6 I=1,NØ
SUMQ=SUMQ+1/SD(I)
SUMA=SUMA+ALPHA(I)/SD(I)
SUMBQ=SUMBQ+BETA(I)/SD(I)
SUMB2Q=SUMB2Q+BETA(I)**2/SD(I)
SUMAB=SUMAB+(ALPHA(I)*BETA(I))/SD(I)
6 CØNTINUE
51 CØNTINUE
ALAMF=((SUMB2Q+1/VAR)*(2*AK-ALBAR*SUMA)+ALBAR*SUMBQ*(
+SUMAB-AM/VAR))/(SUMQ*(SUMB2Q+1/VAR)-SUMBQ**2)
ALAM1=(ALBAR*(AM/VAR-SUMAB)-SUMBQ*ALAMF)/(SUMB2Q+1/VAR)
IF (IØP1.NE.1) GØ TØ 305
PRINT 301,ALAM1,ALAMF
305 CØNTINUE
301 FØRMAT (*LAMBDA1 =*,F10.2,*        LAMBDAF =*,F10.2)
DØ 9 I = 1,NØ
IF (X(I).GT.ALØW) GØ TØ 52
X(I)=ALØW
GØ TØ 9
52 IF (X(I).LT.AHIGH) GØ TØ 53
X(I)=AHIGH
```

```
GØ TØ 9
53 X(I)=(ALBAR*ALPHA(I)+ALAM1*BETA(I)+ALAMF)/SD(I)*0.5
9 CØNTINUE
X(NNØ)=(ALBAR*AM-ALAM1)/VAR*0.5
ICHECK=0
DØ 50 I=1,NØ
IF (X(I).GE.ALØW.AND.X(I).LE.AHIGH) GØ TØ 50
ICHECK=1
SUMQ=SUMQ-1/SD(I)
SUMAB=SUMAB-ALPHA(I)*BETA(I)/SD(I)
SUMA=SUMA-ALPHA(I)/SD(I)
SUMBQ=SUMBQ-BETA(I)/SD(I)
SUMB2Q=SUMB2Q-BETA(I)**2/SD(I)
IF (X(I).LT.ALØW+0.0001) GØ TØ 50
AK=AK-AHIGH
50 CØNTINUE
IF (ICHECK.NE.0) GØ TØ 51
SUM=0
DØ 10 I=1,NØ
SUM=SUM+X(I)
10 CØNTINUE
IF (SUM.LT.1.001) GØ TØ 307
DØ 308 I=1,NØ
308 X(I)=0
SUM=1
SU=0
```

```
AMAX=-100
311 DØ 309 I=1,NØ
IF (E(I).GE.AMAX) LZZ=I
309 IF (E(I).GE.AMAX) AMAX=E(I)
X(LZZ)=AHIGH
AMAX=-100
E(LZZ)=-10000
SU=SU+X(LZZ)
IF (SU.LT.0.99) GØ TØ 311
IF (SU.GT.1.001) X(LZZ)=1+AHIGH-SU
307 DØ 312 I=1,NØ
PØRT(I)=PØRT(I)+100*X(I)/P1(L2,I)
MPØRT(I)=MPØRT(I)+100/(NØ*P1(L2,I))
IF (X(I).GT.0.0.AND.IØP2.EQ.1) PRINT 200,STØCK(I),X(I)
312 CØNTINUE
200 FØRMAT (A4,6F6.4)
VALUEM=0
VALUEP=0
L2=L2+1
DØ 12 I=1,NØ
VALUEM=VALUEM+MPØRT(I)*P1(L2,I)
12 VALUEP=VALUEP+PØRT(I)*P1(L2,I)
IF (SUM.LT.0.99) PRINT,SUM
IF (L1.LT.47) GØ TØ 11
PRINT 202,L1,VALUEP,VALUEM
202 FØRMAT (*PERIØD ENDING #*,I2,/
```

```
+,*PØRT IS*,F9.2,*      MART PØRT IS*,F9.2,/)
 11 CØNTINUE
 STØP
 END
 SUBRØUTINE REGRESS (DJ,A,IDS,AR,AM,AV,AVV)
 DIMENSIØN DJ(IDS),A(IDS)
 SDJ=0
 SS2=0
 SSD=0
 SS=0
 DØ 1 I=1,IDS
 SDJ=SDJ+DJ(I)
 SS2=SS2+A(I)*A(I)
 SSD=SSD+A(I)*DJ(I)
 1 SS=SS+A(I)
 DEL=IDS*SS2-SS*SS
 AR=(SDJ*SS2-SS*SSD)/DEL
 BR=(IDS*SSD-SDJ*SS)/DEL
 AM=0
 AV=0
 DØ 2 I=1,IDS
 2 AM=AM+B(I)
 AM=AR+BR*AM/IDS
 DØ 3 I=1,IDS
 3 AV=SQRT(AV/IDS)
 AVV=AV/AM*100
 RETURN
```

APPENDIX B

THEORY OF MEAN-VARIANCE MODELS

The Mean-Variance portfolio selection models assume that there is a relationship between risk and reward. Risk is usually associated with the variance of the portfolio, while the reward is associated with the expected return of the portfolio. There is some historical basis for this since the larger-variance institutional growth funds tend to outperform the smaller-variance income funds.

E-V REGION

The variance and expected value of each individual security is calculated and plotted as shown in Figure 1.

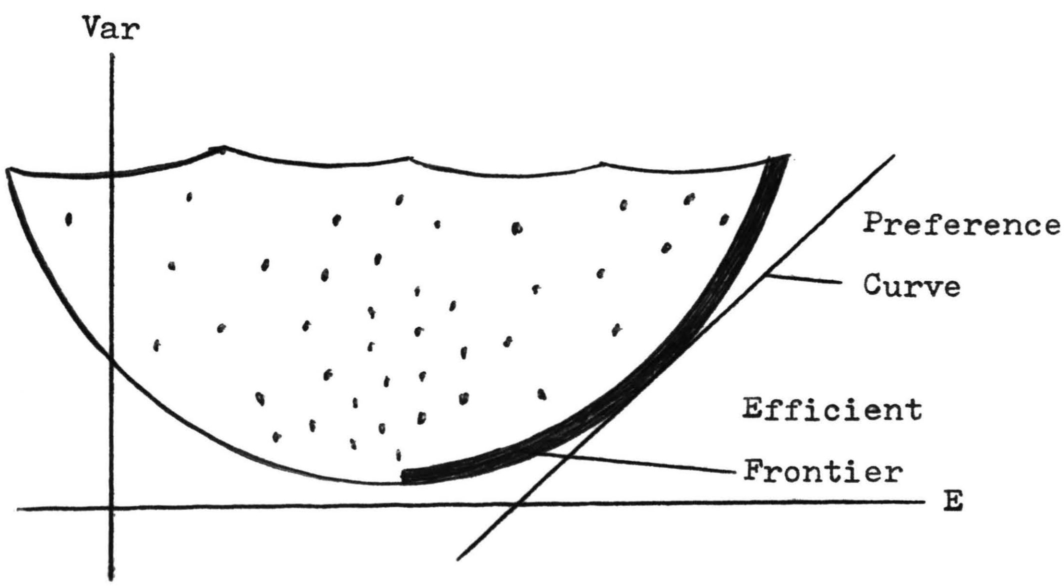

Figure 1.

The bounded region encloses all possible combinations of these securities. This set of feasible portfolios is known as the E-V region.

EFFICIENT FRONTIER

A portfolio is defined to be efficient if for a given level of risk, or variance, the expected return of the portfolio is greater than or equal to the expected return of all other portfolios that have the same level of risk, or variance. A portfolio is also defined to be efficient if for a given level of reward, or expected return, the variance of the portfolio is less than or equal to the variance of all other portfolios that have the same level of reward, or expected return. The set of all efficient portfolios form the lower right boundary of the E-V region as shown in Figure 1.

PREFERENCE CURVE

The preference curve, or indifference curve, can be used to represent investors' feelings toward risk and reward. The preference curve relates the trade-off between the variance and expected return. In particular it is the line

$$V = \lambda E + Z$$

as shown in Figure 1. The larger λ becomes the riskier the portfolio becomes. The optimal Mean-Variance efficient portfolio is the portfolio that occurs where the preference curve is tangent to the efficient frontier. Thus, the problem is to

$$\text{Minimize } Z = -\lambda E + V$$

subject to the constraint that the portfolio must lie on the efficient frontier. By varying λ from infinity to zero, it is possible to generate all portfolios on this frontier.

This portfolio selection model is also referred to as the Efficient-Market model. Sometimes the phrase "capital market theory" is used to denote the same approach.

CRITICAL-LINE ALGORITHM

The algorithm that Markowitz developed to solve this problem is called the Critical-Line algorithm. This algorithm uses the full variance-covariance matrix. It begins by finding the highest-expected-return portfolio. This portfolio is the first corner portfolio. The algorithm then moves down the efficient frontier. As soon as another security enters the current portfolio this portfolio becomes the next corner portfolio. The same is true when a security leaves the current portfolio. Thus, a value of λ is associated with each corner portfolio. The last portfolio selected by the algorithm is the minimum-variance portfolio. This occurs when λ is zero.

One of the characteristics of corner portfolios is that no two adjacent corner portfolios differ by more than one security. This means that all portfolios on the efficient frontier can be represented by linear combinations of corner portfolios.

In practice it is very difficult to apply this algorithm when the full variance-covariance matrix is used. For example, if there are n securities under consideration then an n+3 by n+3 matrix must be inverted in order to find each corner portfolio. In the case of 30 securities it is not uncommon to find 20-40 corner portfolios on the efficient frontier. Each time one of these portfolios is found it requires the inversion of a 33 by 33 matrix. Also, these matrices can be very illconditioned at times and this compounds the problem.

DIAGONALIZED MODEL

In order to overcome the problems presented above, Sharpe reduced the variance-covariance matrix to a diagonal matrix. Sharpe suggested that the return on any security was related to the performance of some index of business activity, such as the Dow Jones Industrial Average. The risk of a security is measured by a standard deviation

σ_i = the standard deviation of e_i

e_i = the rate of return on security i

e_i = the percentage price change

The rate of return on a security is assumed to equal a linear function of the percentage change in the overall market.

$$e_i = a_i + b_i I + c_i$$

where

a_i = some constant

b_i = the market sensitivity of security i

I = the percentage change in the overall market

c_i = the difference between the return on security i and that predicted by its relationship with the overall market

and

$i = 1,2,\ldots,n$

This model is based on several assumptions about the random-error term c_i. Assuming

$$E(c_i) = 0$$

$$\mathrm{Var}(c_i) = q_i$$

$$\mathrm{Cov}(c_i, c_k) = 0 \qquad \text{for } i \neq k$$

and

$$\mathrm{Cov}(c_i, I) = 0$$

then the regression parameters a_i and b_i are unbiased, minimum variance linear estimates of the true regression parameters. The variance of any portfolio is now seen to be dependent only on the variances of the c_i's and I since all of the covariances are zero. Thus, the diagonalized model.

In this dissertation the a_i's and b_i's, which are usually referred to as α's and β's in the literature, were calculated by using a least squares fit on six sample points. For example, $e_{i1}, e_{i2}, \ldots, e_{i6}$ and $I_1, I_2, \ldots, I_6$ represent percentage gains, or losses, for security i and the market during the six two-month intervals of the previous year. See Figure 2. Other time spans were tried but there was no improvement in the performance of the portfolios that were calculated.

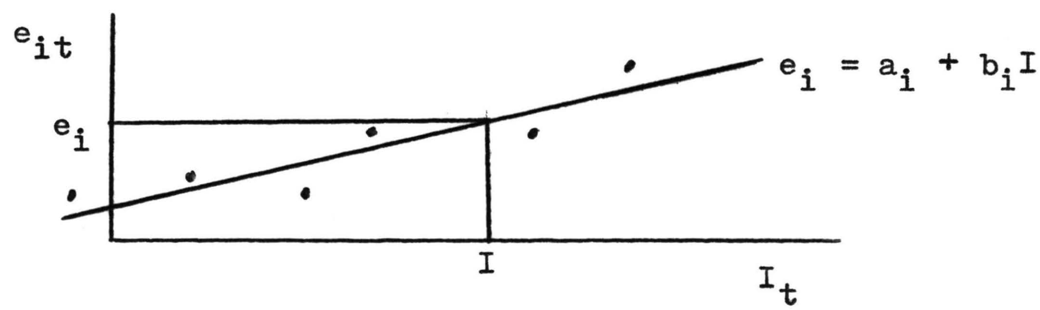

Figure 2.

SIMPLIFIED ALGORITHM

Suppose an investor desires to find the minimum-variance portfolio by using the critical-line algorithm. This means that all corner portfolios must be found. This is very inefficient if only one portfolio is desired. The simplified algorithm was designed to approximate a specific mean-variance efficient portfolio. This eliminates the necessity of calculating unwanted portfolios. The savings in computer time is tremendous. For example, the Critical-Line algorithm was applied to the 30 Dow Industrials with 73 minimum-variance portfolios being found. When the full variance-covariance matrix was used the CPU time totaled approximately two hours. This time was cut to approximately 20 minutes when the diagonalized model was used. The simplified algorithm reduced the time to approximately two minutes.

MARKET PORTFOLIO

The market portfolio consists of all securities weighted evenly in dollar amounts. For example, let x_j represent the amount of capital at decision point j and y_{ij} the amount of capital invested in security i. Then on a percentage basis $v_{ij} = y_{ij}/x_j$, $x_j \neq 0$, represents the relative amount of capital being invested in security i at decision point j. For 30 securities the portfolio is represented by $(v_{1j}, v_{2j}, \ldots., v_{30j})$, or simply $\overline{v}_j$. The market portfolio is the portfolio where all of the v_{ij}'s are equal. In the case of 30 securities this would be $(1/30, 1/30, \ldots., 1/30)$.

VITA

John W. Marsh was born on January 15, 1939, in Cornelius, Oregon. He received his secondary education at Hillsboro Union High School in Hillsboro, Oregon. After attending Walla Walla College in College Place, Washington for one year, he completed his work for the Bachelor of Science degree with a major in Mathematics at Pacific University in Forest Grove, Oregon.

After attending graduate school for a year at Oregon State University in Corvallis, Oregon, he returned to Pacific University as an Instructor of Mathematics and Computer Science. This appointment ran from September 1964 to September 1968. It was during the summer of 1967 that he was granted a N.S.F. fellowship to attend a Computer Science Institute at the University of Missouri at Rolla. He was granted another fellowship for the summer of 1968 and he stayed there for the next three years as a graduate assistant. In 1970 he received his Master of Science degree in Computer Science at the University of Missouri at Rolla. He has been employed by the University of Nebraska at Omaha as an Assistant Professor of Mathematics since September 1971.

On August 10, 1961 he was married to the former Jeri J. Lee of Oakland, Oregon. They have four children, Robby, Randy, Rusty, and Dinah.